Gauged $O(n)$ spin models in one dimension

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Abstract

We consider a gauged $O(n)$ spin model, $n \geq 2$, in one dimension which contains both the pure $O(n)$ and RP^{n-1} spin models and which interpolates between them. Various formulations of this theory are given, one of which shows that it belongs to a general set of models for which it has been suggested that, for $n > 2$, three distinct universality classes exist and that RP^{n-1} and $O(n)$ belong to different classes. We show that our model is equivalent to the non-interacting sum of an $O(n)$ model and the Ising model which allows a simple derivation of the complete mass spectrum that scales in the continuum limit. We demonstrate that there are only two universality classes, one of which contains the $O(n)$ and RP^{n-1} models and the other which has a tunable parameter but which is degenerate in the sense that it arises from the direct sum of the $O(n)$ and Ising models.

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1 Introduction

Recently, there has been a considerable amount of discussion in the literature [1, 2, 3, 4, 5, 6, 7, 8] concerning the nature of the universality classes of models which contain the $O(n)$ spin model and the related RP^{n-1} model. In particular, one-dimensional versions of these theories have been discussed [6, 7, 8] in order to shed light on the mechanism behind the classification of the universality classes and to give an explicit calculation which demonstrate the results. For example, Caracciolo et. al [1, 2] have given numerical evidence to support the statement that in 2D the RP^2 and $O(3)$ spin models have different continuum limits which contradicts the usual belief that the universality classes of these models are the same. In contrast, Niedermayer et. al [4] and Hasenbusch [3] argue that in 2D the $O(3)$ and RP^2 models do indeed belong to the same universality class as long as one is careful to consider the right operators. In support of this conclusion Catterall et. al [5] using the Monte-Carlo renormalization group conclude that the scaling observed numerically in RP^2 and assumed to be associated with a continuum limit is, in fact, only apparent and that the RP^2 model is in the basin of attraction of the $O(3)$ fixed point although very large correlation lengths would be needed to verify this fact.

In 1D Campostrini et. al [7], Cucchieri et. al [6] and Seiler and Yildirim [8] investigate the transfer matrix of general models with $O(n)$ spins and they conclude that, for $n > 2$, there exist three distinct universality classes and that $O(n)$ and RP^{n-1} are not in the same class. This result is cited as support for the similar conclusions in the case of 2D models. In this paper we consider a gauged $O(n)$ model in 1D, $n \geq 2$, similar to the 2D one discussed in [5] and we show that when the gauge field in summed over the resulting action falls in the class of models investigated by the authors cited above. We show that a field redefinition decomposes our model into the non-interacting sum of an $O(n)$ spin model and an Ising model and that by considering an appropriate correlation function the mass ratios R_l computed by Seiler and Yildirim [8], which scale in the continuum limit, follow immediately from this observation. However, it is also evident from the Ising formulation that there are other scaling masses in the theory which are missed in their analysis because they do not directly correspond to eigenstates of the transfer matrix in their formulation. These correspond to the states which contribute to a different correlator in which local operators are connected by a gauge string. In the Ising formulation the transfer matrix is well defined and its eigenstates give rise to the full set of scaling masses characterizing the continuum limit. In any of the formulations susceptibilities can be defined which are sensitive to the complete set of states and which therefore define the continuum limit. The outcome is that there are two universality classes one of which contains the $O(n)$ and RP^{n-1} models and the other which has a tunable parameter. This second class is seen to arise simply from the non-interacting sum of the $O(n)$ and Ising models and so is degenerate in that it is an obvious way of producing an effect of this kind. The introduction of Ising spins is closely related to the approach adopted by Niedermayer et. al [4]

In section 2 we describe the gauge model and its various different manifestations; in section 3 we compute the spectrum; in section 4 we calculate the scaling mass ratios for different choices of continuum limit, and in section 5 we draw our conclusions.

2 The Model

The action is given by

$$S(\mathbf{s},\sigma,\boldsymbol{j},\lambda) \;=\; \sum_x \;\beta f\,(\mathbf{s}_x\cdot\mathbf{s}_{x+1})\,\sigma_{x,x+1} \;+\; \boldsymbol{j}_x\cdot\mathbf{s}_x \;+\; \lambda_{x,x+1}\,\sigma_{x,x+1}\;, \tag{2.1}$$

where $\mathbf{s}_x$ is a unit n-component vector spin field, $\sigma_{x,x+1}$ is a Z_2 gauge field, and $f\,(z)$ is an odd function that has its maximum in $[-1,1]$ at $z=1$ and the derivative $f'(1)>0$ exists. The fields $\boldsymbol{j}_x$ and $\lambda_{x,x+1}$ are sources coupled to the spin and gauge fields respectively. This action is invariant under the gauge transformation

$$\begin{aligned} \mathbf{s}_x &\;\rightarrow\; h_x\,\mathbf{s}_x\;, \\ \sigma_{x,x+1} &\;\rightarrow\; h_x\,\sigma_{x,x+1}\,h_{x+1}\;, \\ \boldsymbol{j}_x &\;\rightarrow\; h_x\,\boldsymbol{j}_x\;, \\ \lambda_{x,x+1} &\;\rightarrow\; h_x\,\lambda_{x,x+1}\,h_{x+1}\;, \end{aligned} \tag{2.2}$$

where the gauge transformation h_x takes values in Z_2 . The partition function is

$$\mathcal{Z} \;=\; \sum_{\{\sigma\}=\pm 1}\int d\{\mathbf{s}\}\;\exp\,(\,S(\mathbf{s},\sigma,\boldsymbol{j},\lambda)\,)\;. \tag{2.3}$$

Here, $\lambda_{x,x+1}$ can be interpreted as a chemical potential for kinks, the 1D equivalent of vortices. The model possesses pure RP^{n-1} symmetry when $\boldsymbol{j}_x=\lambda_{x,x+1}=0,\ \forall x$. In the limit $\boldsymbol{j}_x=0,\ \lambda_{x,x+1}\rightarrow\infty\ \forall x$ the usual $O(n)$ model is recovered. Thus varying $\lambda_{x,x+1}$ allows us to interpolate between these two models and to study the small coupling region of these and the resulting hybrid theories.

In the general theory an important correlator to consider is

$$G(x,y) = \;\langle \boldsymbol{Q}(\mathbf{s}_x)\cdot\prod_{r=x}^{y-1}\sigma_{r,r+1}\cdot\boldsymbol{Q}(\mathbf{s}_y)\rangle\;\Big|_{\boldsymbol{j}=\lambda=0}\;, \tag{2.4}$$

where $\boldsymbol{Q}(\mathbf{s})$ is an odd–parity tensor function of its argument: $\boldsymbol{Q}(-\mathbf{s})=-\boldsymbol{Q}(\mathbf{s})$. The form of this correlator is dictated by the requirement that for RP^{n-1} it must be gauge invariant. Clearly,

$$G(x,y) \;=\; \boldsymbol{Q}\left(\frac{\partial}{\partial\boldsymbol{j}_x}\right)\cdot\frac{\partial}{\partial\lambda_{x,x+1}}\cdots\frac{\partial}{\partial\lambda_{y-1,y}}\cdot\boldsymbol{Q}\left(\frac{\partial}{\partial\boldsymbol{j}_y}\right)\,\log\mathcal{Z}\Big|_{\boldsymbol{j}=\lambda=0}\;. \tag{2.5}$$

If we do the sum over the $\{\sigma\}$ gauge fields explicitly in equation (2.1) we find the action as a function of the $\{\mathbf{s}\}$ only to be

$$A(\mathbf{s},\boldsymbol{j},\lambda) \;=\; \sum_x\;\beta\,g\,(\mathbf{s}_x\cdot\mathbf{s}_{x+1},\lambda_{x,x+1}) \;+\; \boldsymbol{j}_x\cdot\mathbf{s}_x\;, \tag{2.6}$$

with

$$g(z,\lambda) \;=\; \log\left[\,\cosh(\beta\,f(z) \;+\; \lambda)\,\right]\;. \tag{2.7}$$

For $\lambda_{x,x+1}=0,\ \forall x$ we can, instead, consider the RP^{n-1} invariant action $g(z,0)$ as given and then f is determined by the inverse of this procedure:

$$f(z) \;=\; \frac{1}{\beta}\,\mathrm{sign}(z)\cosh^{-1}\left[\exp(\,\beta g(z,0)\right]\;, \tag{2.8}$$

where the positive branch of $\cosh^{-1}$ is chosen. Using equation(2.5) we find

$$G(x,y) \;=\; \langle \boldsymbol{Q}(\mathbf{s}_x) \cdot \prod_{r=x}^{y-1} \omega_{r,r+1} \cdot \boldsymbol{Q}(\mathbf{s}_y) \rangle \;\Big|_{\boldsymbol{j}=\lambda=0} \;, \tag{2.9}$$

where

$$\omega_{x,x+1} = \tanh\left[\, \beta f\,(\mathbf{s}_x \cdot \mathbf{s}_{x+1})\,\right] \;. \tag{2.10}$$

The composite fields $\{\omega\}$ play the rôle of gauge-like fields. We are interested in the physics near to the transition point at zero temperature, i.e., $\beta \to \infty$. In this limit the configurations that contribute appreciably are those for which $\beta\,|f| \gg 1$ and so we have

$$\omega_{x,x+1} \;=\; \tanh(\beta f(\mathbf{s}_x \cdot \mathbf{s}_{x+1})) \;\stackrel{\beta\to\infty}{\longrightarrow}\; \mathrm{sign}(\mathbf{s}_x \cdot \mathbf{s}_{x+1}) \;. \tag{2.11}$$

For example, for $\beta \to \infty$ the action $A(\mathbf{s},\boldsymbol{j},\lambda)$ for pure RP^{n-1} behaves like

$$A(\mathbf{s},\boldsymbol{j},\lambda) \;\sim\; \sum_x \beta\,|f(\mathbf{s}_x \cdot \mathbf{s}_{x+1})| \;. \tag{2.12}$$

The two formulations for the action in equations (2.1) and (2.6) are entirely equivalent with correspondingly equivalent correlators defined in equations (2.4) and (2.9). In the latter case the rôle of the gauge field is played by the link variable $\{\omega\}$ defined in equations (2.10) and (2.11).

For the case of $O(n)$ the eigenfunctions of the transfer matrix are the harmonic functions $Y_l(\mathbf{s})$ on S_{n-1}, and in the special case where $f(z) = z$ the model is the discrete approximation to the quantum mechanics of a particle constrained to S_{n-1} which has eigenfunctions $Y_l(\mathbf{s})$ and associated energy eigenvalues $E_l = l(l+n-2)/4\beta$.

For the general model the transfer matrix can be written as ($\boldsymbol{j}_x = 0, \forall\; x$)

$$T(x,x+1;\mathbf{s},\sigma,\lambda) =$$
$$\exp\,(\lambda_{x,x+1}\sigma_{x,x+1}) \left[1 + \sum_{l\text{ even}} \mu_l(\beta)\; Y_l(\mathbf{s}_x)Y_l(\mathbf{s}_{x+1}) + \sum_{l\text{ odd}} \nu_l(\beta)\; \sigma_{x,x+1}\; Y_l(\mathbf{s}_x)Y_l(\mathbf{s}_{x+1})\right] \;. \tag{2.13}$$

The general set of correlators takes the form

$$F_l(x,y) \;=\; \langle Y_l(\mathbf{s}_x)\; Y_l(\mathbf{s}_y)\rangle \;, \tag{2.14}$$

$$G_l(x,y) \;=\; \langle Y_l(\mathbf{s}_x) \prod_{r=x}^{y-1} \sigma_{r,r+1}\; Y_l(\mathbf{s}_y)\rangle \;. \tag{2.15}$$

For the $O(n)$ model $F_l = G_l$, and in the RP^{n-1} case $G_l = 0$ for l even, and $F_l = 0$ for l odd whenever $x \neq y$.

In formulation without explicit gauge field, such as is given in equation (2.12), the transfer matrix is given (up to an overall irrelevant normalization) by

$$T'(x,x+1;\mathbf{s}) =$$
$$1 \;+\; \sum_{l\text{ even}} \mu_l(\beta)\; Y_l(\mathbf{s}_x)Y_l(\mathbf{s}_{x+1}) \;+\; \tanh(\lambda_{x,x+1}) \sum_{l\text{ odd}} \nu_l(\beta)\; Y_l(\mathbf{s}_x)Y_l(\mathbf{s}_{x+1}) \;, \tag{2.16}$$

and then the correlator G_l takes the form

$$G_l(x,y) = \langle Y_l(\mathbf{s}_x) \prod_{r=x}^{y-1} \omega_{r,r+1}\, Y_l(\mathbf{s}_y)\rangle \ , \tag{2.17}$$

whereas the expression for F_l is unchanged.

The question to be addressed is whether the G_l correlators correspond to the correlation of local operators which interpolate states each labelled by its energy. In the next section we shall show that this is the case and we shall compute the mass ratios which characterize the continuum limit, $\beta \to \infty$.

3 The spectrum

In the previous section we have shown the equivalence between two formulations of the 1D theory which includes both the $O(n)$ and RP^{n-1} models and which interpolates between them. The equivalence between the respective formulations of the relevant correlation functions was also demonstrated. Hence, it is sufficient to determine the spectrum of the model using the version formulated with an explicit gauge field.

We first perform a field redefinition. Define $\mathbf{r}_x$ and ϵ_x by

$$\sigma_{x,x+1} = \epsilon_x \epsilon_{x+1} \ , \qquad \mathbf{r}_x = \epsilon_x \mathbf{s}_x \ . \tag{3.1}$$

This is possible because the field $\sigma_{x,x+1}$ can be always written as a pure gauge in one dimension. We also note that the definition of $\mathbf{r}_x$ corresponds to the general transformation

$$\mathbf{s}_x = \prod_{z=-\infty}^{x-1} \sigma_{z,z+1}\, \mathbf{r}_x \ .$$

Since $f(z)$ is an odd function the action now becomes

$$S(\mathbf{s},\sigma,\boldsymbol{j},\lambda) \ = \ \sum_x \ \beta f\left(\mathbf{r}_x \cdot \mathbf{r}_{x+1}\right) \ + \ \epsilon_x \boldsymbol{j}_x \cdot \mathbf{r}_x \ + \ \lambda_{x,x+1}\, \epsilon_x \epsilon_{x+1} \ . \tag{3.2}$$

The model has decoupled into an $O(n)$ spin model plus an Ising model with inter-site coupling $\lambda_{x,x+1}$. The correlators (2.14) and (2.15) can now be expressed as

$$\begin{aligned} F_l(x,y) \quad &= \quad \begin{cases} \langle Y_l(\mathbf{r}_x)\, Y_l(\mathbf{r}_y)\rangle_O \ , & l \ \text{even} \\ \langle Y_l(\mathbf{r}_x)\, Y_l(\mathbf{r}_y)\rangle_O \ \langle \epsilon_x \ \epsilon_y \rangle_I \ , & l \ \text{odd} \end{cases} \\ G_l(x,y) \quad &= \quad \begin{cases} \langle Y_l(\mathbf{r}_x)\, Y_l(\mathbf{r}_y)\rangle_O \ \langle \epsilon_x \ \epsilon_y \rangle_I \ , & l \ \text{even} \\ \langle Y_l(\mathbf{r}_x)\, Y_l(\mathbf{r}_y)\rangle_O \ , & l \ \text{odd} \end{cases} \end{aligned} \tag{3.3}$$

Where the subscripts O and I signify a correlator in the pure $O(n)$ model and the decoupled Ising model, respectively.

We consider the situation where $\boldsymbol{j}_x = 0, \forall x$ and $\lambda_{x,x+1} = \text{constant} \equiv \lambda$. There is a residual global symmetry of $\epsilon \to -\epsilon$, $\forall x$ which is not in the original model. This symmetry can be accounted for by fixing $\epsilon_{-\infty} = 1$ but is unimportant for $\lambda < \infty$ since the Ising system is disordered. At the transition point, $\lambda = \infty$, this condition fixes the magnetization, and in this particular case the Ising spins are frozen with $\langle \epsilon_x \ \epsilon_y \rangle_I = 1$ and so F_l and G_l are not distinct. There is one set of correlators corresponding to the $O(n)$ model.

As $\beta \to \infty$ a continuum limit is approached and the dominant fluctuations in $\mathbf{s}$ are controlled by $f(z)$ near to $|z| = 1$. So, since $f(z)$ is an odd function, the important contributions to the partition function can be calculated for the model in which we make the replacement

$$f(z) \to f'(1)z \ . \tag{3.4}$$

This factor $f'(1)$ can be absorbed into redefinition of β: $\beta' = f'(1)\beta$, and so in the continuum limit, from equation (3.2) and without loss of generality, the spectrum of the $O(n)$ part of the model coincides with the spectrum of the quantum mechanics of a particle constrained to S_{n-1} as was discussed as a special case in the previous section. Since we are interested in mass ratios as the continuum is approached we confine the discussion to the model where $f(z) = z$ and drop the prime on β, although we emphasize that this is not a special case but applies to all actions of the form given in equation (2.1) as the continuum limit is approached.

The interpolating states have wavefunctions taking values on the direct product space $S_{n-1} \otimes Z_2$, where the Z_2 is the space associated with the Ising degrees of freedom. The full spectrum of masses is therefore

$$\begin{aligned} m_l &= l(l+n-2)/4\beta \ , \\ M_l &= \frac{l(l+n-2) \ + \ 4c(\lambda)}{4\beta} \ , \end{aligned} \tag{3.5}$$

where we define $c(\lambda) = \beta \log(\coth(\lambda)) \Rightarrow \lambda = \frac{1}{2}\log(\coth(c/2))$. The mass spectrum of the interpolating states for F_l (G_l), which we denote μ_l^F (μ_l^G), is then m_l for l even (odd) and M_l for l odd (even).

For states to survive in the continuum limit their associated masses must vanish in this limit: all other states are lattice artifacts. For both m_l and M_l this will only happen as $\beta \to \infty$, as expected, but for M_l we also require that in this limit λ behaves such that $c(\lambda)$ is finite or diverges with β more slowly than $\mathrm{O}(\beta)$. It is the behaviour of $c(\lambda)$, and hence of λ, that distinguishes between the different kinds of continuum theory obtained in the limit $\beta \to \infty$, and this will lead to a classification of the universality classes that are possible.

4 The continuum limit

To probe the continuum limit we can use the spectrum obtained from F_l and G_l above. In [8] the authors discuss the transfer matrix of models which include the models considered here when they have been recast into the form given in equations (2.6) and (2.7). Whilst there is not a one-to-one correspondence we shall show that our models give results which correspond to each of the universality classes cited in [8] and so, by universality, our results are pertinent to the gamut of models considered in this reference. From the discussion in section 2 and equations (2.14) and (2.17) we see that to access all possible states in the case where the gauge field has been summed over we must introduce the "gauge string" formed from the $\omega_{x,x+1}$. In 1D this will still give rise to local interpolating operators. To just consider the transfer matrix and omit these degrees of freedom corresponds to just discussing the spectrum of states contributing to F_l but not G_l. This is the case for the analysis in [8]. To facilitate the first part of

the discussion we take λ to depend on β so that as $\beta \to \infty$

$$
\begin{array}{lcl}
 & c & \sim \; c_0\beta^{\eta} \; , \\
\Longrightarrow & & \\
 & \lambda(\beta) & \sim \; \begin{cases} \frac{1}{2}\left((1-\eta)\log(\beta)+\log(2/c_0)\right) \; , & \eta < 1 \\ \frac{1}{2}\log(\coth(c_0/2) \; , & \eta = 1 \\ \exp(-c_0\beta^{\eta-1}) \; , & \eta > 1 \end{cases}
\end{array}
\tag{4.1}
$$

where c_0 is a constant. This is not the most general parametrization but it suffices to elucidate the important cases and the conclusions will not be changed by using a more general form. For $\eta < 1$ the Ising states survive into the continuum limit, but for $\eta \geq 1$ they are lattice artifacts only and are not relevant as $\beta \to \infty$ and so we must recover the spectrum of the $O(n)$ model in the limit. The dependence of the behaviour of $\lambda(\beta)$ on η is significant and takes the form

$$
\begin{array}{ll}
\eta < 1 & \lim_{\beta\to\infty} \lambda(\beta) = \infty \; , \\
\eta = 1 & \lim_{\beta\to\infty} \lambda(\beta) = \frac{1}{2}\log(2/c_0) \; , \\
\eta > 1 & \lim_{\beta\to\infty} \lambda(\beta) = 0 \; .
\end{array}
\tag{4.2}
$$

Consequently, for $\lambda = 0$ we must recover the spectrum of the $O(n)$ model in the continuum limit. This shows that the RP^{n-1} model, for which $\lambda = 0$, has the same continuum limit as the $O(n)$ model which has $\lambda = \infty$. We shall further elucidate this result below.

We consider the continuum limit mass ratios $R_l = \mu_l^F/\mu_1^F$ and identify the different classes:

$$
R_l = \begin{cases} \frac{l(l+n-2)}{n-1} \; , & \eta < 0 \; , \quad c \to 0 \quad \textbf{I} \\[2ex] \frac{l(l+n-2)\,+\,4c_0\,(1-(-1)^l)/2}{(n-1)+4c_0} \; , & \eta = 0 \; , \quad c \to c_0 \quad \textbf{II} \\[2ex] \frac{1-(-1)^l}{2} \; . & 0 < \eta < 1 \; , \quad c \to \infty \quad \textbf{III} \end{cases}
\tag{4.3}
$$

These mass ratios are identical to those quoted for classes **I, II, III** in reference [8].

We could similarly discuss the ratios μ_l^G/μ_1^G associated with correlators G_l but it is better to consider the complete spectrum $\{m_l, M_l\}$ given by F_l and G_l and define the mass ratios to be $P_l^{(F)} = \mu_l^F/m_1$ and $P_l^{(G)} = \mu_l^G/m_1$. The nature of the continuum limit is determined by considering the P_l rather than the R_l which are incomplete. We then find the three classes above to be characterized by

$$
P_l^{(F/G)} = \begin{cases} \frac{l(l+n-2)}{(n-1)} \; , & \eta < 0 \; , \quad c \to 0 \quad \textbf{I} \\[2ex] \frac{l(l+n-2)\,+\,4c_0\,(1\mp(-1)^l)/2}{(n-1)} \; , & \eta = 0 \; , \quad c \to c_0 \quad \textbf{II} \\[2ex] \frac{l(l+n-2)}{(n-1)} \begin{pmatrix} l \text{ odd} \\ l \text{ even} \end{pmatrix} \; , \quad \infty \begin{pmatrix} l \text{ even} \\ l \text{ odd} \end{pmatrix} \; , & 0 < \eta < 1 \; , \quad c \to \infty \quad \textbf{III} \end{cases}
\tag{4.4}
$$

A ratio of ∞ means that the corresponding state is not in the spectrum.

The nature of the three classes is now clear. Classes **I** and **III** have the same values for the P_l: in **I** the Ising spins are frozen, as remarked earlier, and in **III** the Ising mass gap scales but with infinite ratio to the lowest lying states. In both cases we recover the universality class of the pure $O(n)$ spin model. For class **II** the P_l are labelled by the continuous parameter c_0 which Seiler and Yildirim take to label the continuum of universality classes.

It remains to discuss the cases with $\eta \geq 1$. In these cases the mass μ_1^F does not survive into the continuum and so the ratios R_l have no meaning. From equation (4.2) we have that $0 \leq \lambda < \infty$ and the result for the P_l falls into class **III** of equation (4.4). We conclude that all models with finite non-negative λ belong to the $O(n)$ universality class as was stated earlier. In particular, the RP^{n-1} model belongs to the $O(n)$ universality class. This result is trivially extended to cover all λ, $-\infty \leq \lambda < \infty$.

The universality class **I** and **II** are associated with the fixed point at $(\beta = \infty, \lambda = \infty)$, and the parameter c_0 controls the way in which the trajectories of **II** approach this point.

5 Conclusions

We have considered a class of gauged $O(n)$ models in one dimension, equation (2.1), which includes RP^{n-1}. The gauge field can be summed over and the resultant alternative form for the action, equation (2.6), falls in the class of actions considered by Seiler and Yildirim [8], Campostrini et. al [7] and Cucchieri et. al [6]. The conclusion of these cited works concerning such actions is that there exist three distinct universality classes and that, in particular, RP^{n-1} is not in the same class as $O(n)$. In this paper we have shown that gauged $O(n)$ version of the model can be analyzed in a very straightforward manner and that the scaling mass ratios, R_l, quoted by Seiler and Yildirim can be clearly and simply derived, equation (4.3), and their origin elucidated to be from the non-interacting sum of an $O(n)$ and Ising model. Cucchieri et. al note briefly that Ising degrees of freedom are present in these models but do not comment further on their remark.

The analysis of Seiler and Yildirim, Campostrini et. al, and Cucchieri et. al is based on the transfer matrix formulation of the theories which coincides with the analysis of the correlators F_l alone, equation (2.14), and the masses of the corresponding interpolating states. However, the universality class of a model is described by the properties of the model as the critical point is approached and, in particular, how various susceptibilities diverge. The susceptibilities are defined in terms of the derivatives of the free energy with respect to external fields and it is clear from either version of the model we have studied that the correlators G_l legitimately define a susceptibility through equations like equation (2.5). However, the form for G_l given in (2.9) shows that the masses of the interpolating states will not be determined by an analysis of the transfer matrix (2.16) relevant to the model after the gauge field has been summed over. For the gauge model the transfer matrix in the Ising formulation is simply

$$T_I(x, x+1, \mathbf{s}) \;=\; \exp\left(\beta\, \mathbf{r}_x \cdot \mathbf{r}_{x+1}\right) \begin{pmatrix} \exp(\lambda_{x,x+1}) & \exp(-\lambda_{x,x+1}) \\ \exp(-\lambda_{x,x+1}) & \exp(\lambda_{x,x+1}) \end{pmatrix}, \qquad (5.5)$$

and the state basis is then spanned by

$$Y_l(\mathbf{s}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \qquad Y_l(\mathbf{s}) \begin{pmatrix} 1 \\ -1 \end{pmatrix} . \tag{5.6}$$

In the non-gauge formulation it is not obvious that the $G_l(x, y)$ decay exponentially as $|x - y| \to \infty$ but we have shown that they do possess this property and that the masses of the corresponding states scale into the continuum limit. When the complete mass spectrum is taken into account we obtain the scaling mass ratios P_l in equation (4.4) which show that the universality classes **I** and **III** are the same. In particular, RP^{n-1} belongs to the same universality class as $O(n)$. The existence of a continuous set of universality classes of type **II** is seen to be degenerate in the sense that it arises from the non-interacting sum of two independent models – a construction which is always available to produce such a result.

The results of this paper confirm in one-dimension the results obtained by Niedermayer et. al [4] and Hasenbusch [3] in their study of similar models in two dimensions. Niedermayer et. al also introduce an Ising degree of freedom to reach their conclusions.

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